Area By Double Integration

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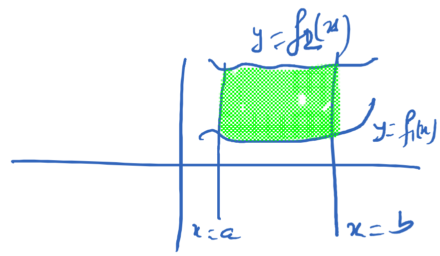
① In Cartesian Co-ordinate

$y = f_1(x)$, $y = f_2(x)$ and the lines are $x = a$, $x = b$

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

dx dy



② Polar Coordinates The Area of the region bounded by the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ and the lines $\theta = \alpha$, $\theta = \beta$

Then $$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

Q:-1 Find the area between the parabola $y^2 = 4ax$ and $x^2 = 4ay$

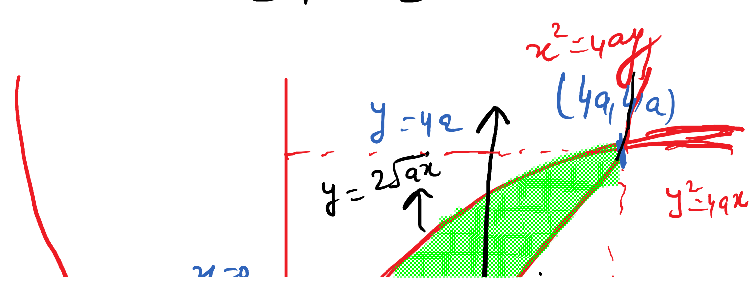
Sol:- Given two curves

$y^2 = 4ax \rightarrow$ Right handed parabola. with vertex (0,0) and Latus rectum = 4a

and $x^2 = 4ay \rightarrow$ upward parabola with vertex (0,0) Latus rectum = 4a.

Intersection of these parabolas:

$y^2 = 4ax$



$$y^2 = 4ax$$

$$\Rightarrow x = \left(\frac{y^2}{4a}\right)$$

$$x^2 = 4ay$$

$$\left(\frac{y^2}{4a}\right)^2 = 4ay$$

$$\frac{y^4}{16a^2} = 4ay$$

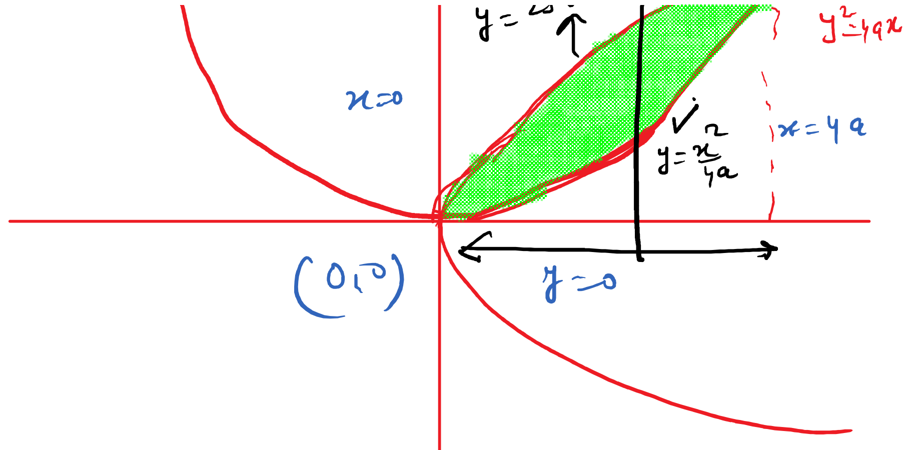
$$y^4 = 64a^3 y$$

$$y^4 - 64a^3 y = 0$$

$$y(y^3 - (4a)^3) = 0$$

$$y = 0, \quad y^3 = 64a^3$$

$$y = 4a$$



When $y = 0$, $x = 0$ $\quad (0,0)$

$y = 4a \quad x = 4a \quad (4a, 4a)$

(x varies from 0 to $4a$
y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$)

or

y varies from 0 to $4a$
x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$

$$\therefore \text{Required Area} = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$

$$= \int_0^{4a} (y)_{x^2/4a}^{2\sqrt{ax}} dx = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a} \left| \frac{x^{3/2}}{3/2} \right|_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)_0^{4a}$$

$$(2\sqrt{a})\left(\frac{2}{3}\right)(4a)^{\frac{3}{2}} - \frac{1}{4a}\left(\frac{(4a)^3}{3}\right)$$

$$\left(2\sqrt{a}\right)\left(\frac{2}{3}\right)(4a)^{\frac{3}{2}} - \frac{1}{4a}\left(\frac{(4a)^3}{3}\right)$$

$$= \frac{4}{3}\sqrt{a}\,(4a)(2\sqrt{a}) - \frac{(4a)^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16}{3}a^2$$

Ans

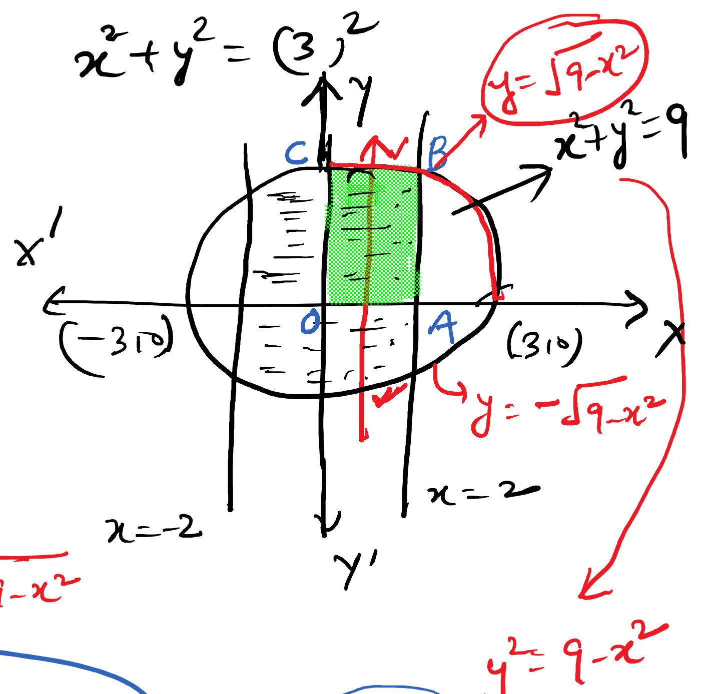
Q:-2 Find the area bounded by the lines $x=-2$, $x=2$ and circle $x^2+y^2=9$.

Sol:- Given equation of circle $x^2+y^2=9$

$x^2+y^2=(3)^2$

$x=-2$, to $x=2$

i.e



Here x varies from -2 to 2

y varies from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$

$\therefore$ Reqd Area $= \displaystyle\int_{-2}^{2}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

$4\displaystyle\int_0^2\int_0^{\sqrt{9-x^2}} dy\,dx$

$-\sqrt{9-x^2}$

or

Reqd Area $= 4\,ar(\triangle ABC) = 4\int_0^2\int_0^{\sqrt{9-x^2}} dy\,dx$

$= 4\int_0^2 (y)_0^{\sqrt{9-x^2}} dx$

$= 4\int_0^2 \sqrt{9-x^2}\,dx$

$\int \sqrt{a^2-x^2} = \left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)$

$= 4\left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right)_0^2$

$= 4\left[\frac{2\sqrt{5}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{2}{3}\right)\right]$

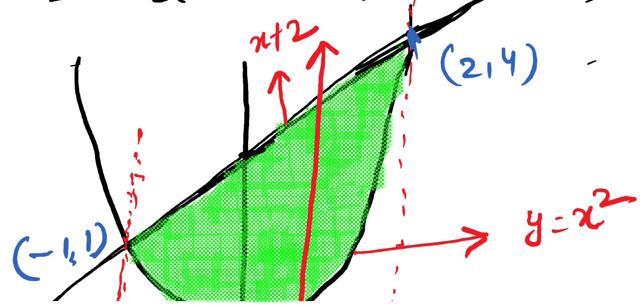
$= 4\sqrt{5} + 18\sin^{-1}\left(\frac{2}{3}\right)$ Ans

Q:- find the area bounded by parabola $y = x^2$ and line $y = x+2$

Sol:- Area bounded by the $y = x^2$ and $y = x+2$

$y = x^2$ is the upward parabola with vertex $(0,0)$

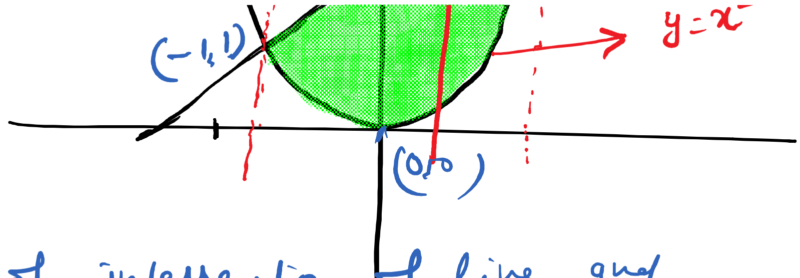
and $y = x+2$ is a Straight line



Put $x = 0$ $y = 2$ $(0,2)$

Put $x = 0$, $y = 2$ $(0, 2)$

Put $y = 0$, $x = -2$ $(-2, 0)$



Now we find out point of intersection of line and parabola

$$x + 2 = x^2$$

$$x^2 - x - 2 = 0$$

$$x = -1, 2$$

when $x = -1$, $y = +1$

$x = 2$, $y = 4$

x varies from -1 to 2

y varies from x^2 to $x+2$

$$\therefore \text{Reqd Area} = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx$$

$$= \int_{-1}^{2} [y]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right)_{-1}^{2}$$

$$= \frac{9}{2}.$$

Change of variables in Double Integral

If A region of xy plane is mapped to the region B of uv plane with transformation

$$x = \phi(u,v)$$
$$y = \psi(u,v)$$

Then

$$\iint_A f(x,y)\,dx\,dy = \iint_B f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv$$

where $|J| = \dfrac{\partial(x,y)}{\partial(u,v)}$

In polar Co-ordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$|J| = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$\iint_A f(x,y)\,dx\,dy = \iint_{A'} f(r,\theta)\,r\,dr\,d\theta$$

Change of variables in a Triple Integral

changing in Spherical polar Co-ordinates.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$$

where $|J| = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \iiint_{V'} f(r, \theta, \phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

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